

Kac-Moody groups over ultrametric fields

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The Kac-Moody groups studied here are the minimal (=algebraic) and split ones, as introduced by J. Tits in [8]. When they are defined over an ultrametric field, it seems natural to associate to them some analogues of the Bruhat-Tits buildings.

Actually I came to this problem when I was trying to build new buildings of non-discrete type. If G is a Kac-Moody group over an ultrametric field K , I was able to build a *microaffine* building \mathcal{I}^μ on which $G(K)$ acts [5]. This building is an union of apartments in one to one correspondence with the maximal split tori and the usual axioms of buildings are satisfied, among them the fundamental axiom: any two points are in a same apartment. It is closely related to the Satake (or polyhedral) compactification of the Bruhat-Tits building of a semi-simple group over K . One knows that this compactification is the disjoint union of the Bruhat-Tits buildings of the semi-simple quotients of all parabolic subgroups of this semi-simple group. For a Kac-Moody group the same definition gives the microaffine building, but now the parabolic subgroups give something in \mathcal{I}^μ only when they are of finite type, so G itself gives nothing. We just have to define the apartments and prove the usual axioms of buildings, see [5].

Unfortunately \mathcal{I}^μ seems to give only a few informations about the structure of $G(K)$. Moreover P. Littelmann asked me whether it could be used to generalize his results with S. Gaussent in the semi-simple case [2]: they proved in particular that a LS-path may be seen (in an apartment of a Bruhat-Tits building over the field of Laurent series $\mathbb{C}((t))$) as an image of a segment of the building under some fixed retraction (with center a sector-germ), satisfying also some numerical condition. It was soon clear that, in the Kac-Moody case, \mathcal{I}^μ is not suitable. One has to mimic more closely the Bruhat-Tits construction. The normalizer of the standard maximal split torus in $G(K)$ acts on the corresponding apartment \mathbb{A} by a group of affine transformations, generated by reflections on walls. But there is a lot of walls (infinitely many directions), moreover in the loop group situation, H. Garland in [1] had proved that there is no Cartan decomposition, so the expected building would not satisfy the fundamental axiom of buildings: it seemed at first too ugly.

Nevertheless it is possible to build this close analogue to Bruhat-Tits buildings for some split Kac-Moody groups (joint work with Stéphane Gaussent):

Theorem. [3] *Suppose that $K = \mathbb{C}((t))$ (or more generally that \mathbb{C} is in the residue field of K) and moreover that G is symmetrizable. Then there exists a set \mathcal{I} , with an action of $G(K)$, containing a subset identified with \mathbb{A} . The stabilizer of \mathbb{A} is the normalizer of the standard maximal split torus and the induced action on \mathbb{A} is as described above. The set \mathcal{I} is covered by the apartments i.e. the conjugates of \mathbb{A} by elements of $G(K)$.*

The unfortunate restriction on K is due to heavy technical complications; more general cases should be proved in the near future.

The space \mathcal{I} doesn't satisfy the fundamental axiom of buildings, so this ugly \mathcal{I} is called a *hovel*. But Iwasawa decomposition is still verified in the Kac-Moody group acting on \mathcal{I} , so any point and any sector-germ in \mathcal{I} are always in a same apartment. This enables us to define a retraction ρ of \mathcal{I} on an apartment with center a sector-germ in this apartment. With this retraction it is possible to prove, for Kac-Moody groups, the above quoted result of S. Gaussent and P. Littelmann about LS-paths and to associate to such a path a quasi-affine toric variety (complex and infinite dimensional) which is a reasonable generalization of the Mirkovic-Vilonen cycles, see [3].

Actually this hovel is not so ugly: we proved in [3] that there is on \mathcal{I} a preorder relation which induces on each apartment the preorder given by the Tits cone. Moreover the sets of increasing (resp. decreasing) segment-germs of origin $x \in \mathcal{I}$ are twin buildings: the residue of \mathcal{I} in x .

There is also an abstract definition of affine hovels in the spirit of the abstract definition of affine buildings given by J. Tits in [7]: to be short the fundamental axiom is now that any point and any sector-germ or any two sector-germs have to be in a same apartment. This abstract definition is satisfied by the hovels constructed for Kac-Moody groups and there are interesting consequences. These affine hovels look like affine buildings, but the spherical buildings associated to affine buildings are replaced by twin buildings. More precisely we recover at infinity some buildings: the parallel classes of sector-faces are the faces of a twin building and the germs of these sector-faces are the points of two microaffine buildings. In the case of the hovel associated to a Kac-Moody group G , these buildings are the, now well known, twin building of $G(K)$ [4] and two microaffine buildings as in [5], one for each of the two possible choices (positive or negative) of the Tits cone in an apartment. The preorder relation and the structure of the residues are also consequences of this abstract definition. See [6].

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